ABSTRACT

Vibration analysis for complicated structures, or for problems requiring large numbers of modes, always requires fine meshing or using higher order polynomials as shape functions in conventional finite element analysis. Since it is hard to predict the vibration mode a priori for a complex structure, a uniform fine mesh is generally used which wastes a lot of degrees of freedom to explore some local modes. By the present wavelets element approach, the structural vibration can be analyzed by coarse mesh first and the results can be improved adaptively by multi-level refining the required parts of the model. This will provide accurate data with less degrees of freedom and computation.

The scaling functions of B-spline wavelet on the interval (BSWI) as trial functions that combines the versatility of the finite element method with the accuracy of B-spline functions approximation and the multiresolution strategy of wavelets is used for frame structures vibration analysis. Instead of traditional polynomial interpolation, scaling functions at the certain scale have been adopted to form the shape functions and construct wavelet-based elements. Unlike the process of wavelets added directly in the other wavelet numerical methods, the element displacement field represented by the coefficients of wavelets expansions is transformed from wavelet space to physical space via the corresponding transformation matrix.

To verify the proposed method, the vibrations of a cantilever beam and a plane structures are studied in the present paper. The analyses and results of these problems display the multi-level procedure and wavelet local improvement. The formulation process is as simple as the conventional finite element method except including transfer matrices to compute the coupled effect between different resolution levels. This advantage makes the method more competitive for adaptive finite element analysis. The results also show good agreement with those obtained from the classical finite element method and analytical solutions.

INTRODUCTION

Today the finite element method is one of the mainly used numerical procedures for structural analysis. Its applications to both static and dynamic problems are widespread. Although its versatility makes it as the general tool in present approaches to structural design, there is a growing concern about the accuracy and reliability of the method. The h, p and h-p finite element extensions discussed in Szabo and Babuska (1991) are procedures which address the above issues.

The accuracy of the h-extension is achieved by refining the mesh while using the same low order polynomials as shape functions. The p-extension uses a fixed mesh while the desired approximate result is promoted by increasing the order of polynomials in the elements. For smooth solution problems, p-extension is converged rapidly. But the presence of singularities may decrease the rate of convergence significantly. This may be overcome by adding h refinement. In addition to the low convergence rate of h-extension, the data management for the multi-level refining mesh is also complicated. In contrast to the h-extension, the p-extension needs much more computational work to construct the local matrices and the order of polynomials may be limited by the condition of the system equations. Although h-p extension is more robust, the remeshing required is still an unavoidable disadvantage.

In this paper, an alternative method is used by employing the B-spline wavelets expansions as the shape function for finite element analysis with a fixed mesh. The wavelets expansions approximation for any function discussed in the introducing
text of Chui (1992) is based on the multiresolution analysis (MRA) which provides a good convergence condition. The improvement for accuracy and reliability can be enhanced by adding higher level wavelets in the required elements without affecting the original modeling. In Briggs and Henson (1993), they suggest that wavelets have built-in multigrid property which is adequate for multi-level problem solving.

The set of wavelet bases is composed by translation and dilation or compression of a compact supported function which is adapted for local gradient variation refinement; see Strang (1989). Furthermore, the spline functions provide excellent approximation with least degrees of freedom (DOFs) for smooth functions. All the above features introduce the proposed method as an attractive finite element extension without mesh modifying.

One of the pioneer numerical experiments of wavelets bases to differential equations was explored by Glowinski, Lawton, Ravachol, and Tenenbaum (1989). They used wavelets expansions as the trial functions in Galerkin method.

The one dimensional numerical examples in their paper suggest that wavelets bases have great potential in numerical application due to its local approximation and multiresolution properties. Although the orthonormal wavelets proposed by Daubechies (1989) were employed in their work, the orthogonality with respect to the square integrable L^2( \mathbb{R} ) space is not suitable for general implementation to finite domain, two transform matrices are constructed to clear the density of the material, and \( \mathcal{C} \) denotes the area of the cross section.

Another characteristic is the uniform translation of the splines bases. According to this feature, the element domain can be divided uniformly into segments. In each segment, the variables are approximated by pieces of contiguous splines as shape functions. The elemental matrices can be obtained by assembling the segmental matrices computed as conventional finite element works. To apply the multi-level solving for finite element analysis, the meshed model is first approximated by spline functions in each element. Then the model is improved by adding higher level B-spline wavelets in the elements which are not accurate enough. This results in an easy way to treat the element connectivity and preserve the original system matrices unchanged by a resolution refinement procedure.

Vibration analysis for complicated structures, or for problems requiring large numbers of modes, always requires fine meshing or using higher order polynomials as shape functions in conventional finite element analysis. Since it is hard to predict the vibration mode a priori for a complex structure, a uniform fine mesh is generally used which wastes a lot of degrees of freedom to explore some local modes. By the present wavelets element approach, the structural vibration can be analyzed by coarse mesh first and the results can be improved adaptively by multi-level refining the required parts of the model. This will provide accurate data with less degrees of freedom and computation. To verify the proposed method, the vibrations of a cantilever beam and a plane structures are studied in the present study. The analyses and results of these problems display the multi-level procedure and wavelet local improvement. The formulation process is as simple as the conventional finite element method except including transfer matrices to compute the coupled effect between different resolution levels. This advantage makes the method more competitive for adaptive finite element analysis. The major contribution of the present study is from the work done by Xiang et al. (2007).

The model problem

To evaluate the implementation of finite element method by the wavelets approximation, the free vibration of frame structures are analyzed as examples. The governing equations in each frame element derived from Magrab (1979) are listed as

\[ EAu'' - \rho A \omega^2 u = 0 \]  \hspace{1cm} (1)

\[ C\ddot{\varphi} - \rho I \omega^2 \varphi = 0 \]  \hspace{1cm} (2)

\[ EI_i \dddot{w_i} + \rho A \omega^2 w_i = 0 \quad i = 1,2 \]  \hspace{1cm} (3)

Here, \( u \) is the element axial displacement, \( \varphi \) denotes the element torsional displacement and \( w_i \) (\( i = 1,2 \)) are the element lateral displacements of bending in two principal planes of the cross section.

The axial, torsional and bending rigidities are represented by \( E, C \) and \( EI \), respectively. \( E \) is the modulus of elasticity, \( \rho \) is the density of the material, \( A \) denotes the area of the cross section, \( I \) is the polar moment of inertia per unit length, (')
represents the differentiation with respect to axial space coordinate of the element and $\omega$ is the natural radial frequency. Since the above equations are uncoupled, the finite element formulation of each vibration type can be treated separately. Here, only the bending equation in one principal direction is written down representatively. The others can be derived in a similar procedure. The elemental bending stiffness and mass matrices $k_e$ and $m_e$ are manipulated respectively as

$$
k_e = \int_a^b \left( \frac{\partial^2 [N(x)]^T}{\partial x^2} \right) dx \left( E I \frac{\partial^2 [N(x)]}{\partial x^2} \right) dx$$

$$
m_e = \int_a^b \rho A[N(x)]^T [N(x)] dx$$

Where $x$ is the axial coordinate of the element, $[N(x)]$ is composed by the shape functions for the displacement.

**B-spline wavelet on the interval $[0,1]$**

B-splines for a given simple node sequence can be constructed by taking piecewise polynomials between the nodes and joining them together at the nodes in such a way as to obtain a certain order of overall smoothness. B-splines of order $m$ are in $C^{m-2}$. Since the function $f(x)$ on the interval $[a, b]$ can be transferred to $[0,1]$ by the transformation formula $\xi = (x-a)/(b-a)$, it only needs to construct $m^{th}$ order B-spline space on the interval $[0,1]$. Generally, the interval $[0,1]$ can be divided into $2^j$ ($j \in \mathbb{Z}^+$ is the scale) segments, and then increasing $m$ nodes outside each endpoint and looking the two $m - 1$ nodes as multiple nodes.

Let $\{x_k^j\}_{k=-m+1}^{2^j+1}$ be a node sequence with $m$-1 multiple nodes at 0 and 1, then the whole node number is $2^j + 2m - 1$, and the node sequence form B-spline functions, which can be further constructed to the $m^{th}$ order nested B-spline subspace $V_{m}^{[0,1]}$. Its basis functions are given below

$$B_{m,k}^j(x) = N_m(2^j \xi - k) \quad (k = -m + 1, ..., 2^j - 1)$$

$$\text{supp} \; B_{m,k}^j = [\xi_k^j, \xi_{m+k}^j]$$

where $N_m(x)$ is cardinal splines. Let $\Phi_{m,k}^j(\xi) = B_{m,k}^j(\xi)$ be the scaling functions of BSPI, we can obtain the multi-resolution analysis (MRA) on the interval $[0,1]$ Chui and Quak (1992). The smoothness order of scaling functions $\Phi_{m,k}^j(\xi)$ is $m - 1$. The support of the inner (without multiple nodes) B-splines occupies $m$ segments and that of the corresponding semi-orthogonal (SO) wavelet occupies $2m-1$ segments.

At any scale $j$, the discretization step is $1/2^j$ which, for $j > 0$, gives $2^j$ segments on $[0,1]$. Therefore, to have at least one inner wavelet on the interval $[0,1]$, the following condition must be satisfied (Goswami, Chan and Chui 1995),

$$2^j \geq 2m - 1$$

While $0$ scale $m^{th}$ order B-spline functions and the corresponding wavelets are given by Chan and Chui (1995), $j$ scale $m^{th}$ order BSWI (simply denoted as BSWI$_m$) scaling functions $\Phi_{m,k}^j(\xi)$ and the corresponding wavelets $\Psi_{m,k}^j(\xi)$ can be evaluated by the following formulas (Xiang et al. 2007),

$$\Phi_{m,k}^j(\xi) = \Phi_{m,k}^j(2^{-j} \xi)$$

$$\begin{cases}
(0 \text{ boundary scaling functions}) & k = -m + 1, \ldots, -1 \\
\phi_{m,2^{j-m-k}}^j(1 - 2^{-j} \xi), & (1 \text{ boundary scaling functions}) k = 2^j - m + 1, \ldots, 2^j - 1 \\
\phi_{m,a}^j(2^j \xi - 2^{-j}k) & (\text{inner scaling functions}) k = 0, \ldots, 2^j - m + 1 
\end{cases}$$

$$\Psi_{m,k}^j(\xi) = \Psi_{m,a}^j(2^{-j} \xi),$$

$$\begin{cases}
(0 \text{ boundary wavelets}) & k = -m + 1, \ldots, -1 \\
\psi_{m,2^{j-2m-k}+1}^j(1 - 2^{-j} \xi), & (1 \text{ boundary wavelets}) k = 2^j - 2m + 2, \ldots, 2^j - m \\
\psi_{m,a}^j(2^j \xi - 2^{-j}k) & (\text{inner wavelets}) k = 0, \ldots, 2^j - 2m + 1 
\end{cases}$$

The wavelets compactly supported intervals are

$$\text{supp} \; \Psi_{m,k}^j(\xi) = \begin{cases}
[0, (2m - 1 + k)2^{-j}], & (0 \text{ boundary wavelets}) \\
[k2^{-j}, 1] & (1 \text{ boundary wavelets}) \\
[k2^{-j}, (2m - 1 + k)2^{-j}] & (\text{inner wavelets}) 
\end{cases}$$

Let $j_0$ be the scale for which the condition Eq. (8) is satisfied. Then for each $j > j_0$, we can get the scaling and wavelet functions easily through Eqs. (9) and (10). Generally, there are $m - 1$ boundary scaling functions and wavelets at 0 and 1, $2^j - m + 1$ inner scaling functions, and $2^j - 2m + 2$ inner wavelets. Figure 1 shows all the scaling functions and wavelets for $m = 4$ at the scale $j = 3$. 
Euler beam-bending element

In order to satisfy the compatibility of unknown field function $w$ and its derivative $dw/dx$ among neighboring elements, each edge node of element DOFs in physical space should include the DOFs of $w$ and its derivative $dw/dx$. Each inner node needs only $w$. The layout of element nodes is shown in Fig. 2, the edge nodes are 1 and $n+1$, and the inner nodes are 2, 3, $\ldots$, $n$. The coordinate values are

$$x_h \in [x_1, x_{r+1}] \quad (1 \leq h \leq r + 1) \quad (12)$$

Using the relation

$$\xi = \frac{x-x_1}{l_e} \quad (1 \leq \xi \leq 1) \quad (13)$$

we would obtain the mapping value $x_h$ of each node coordinate value $x_k$ as

$$\xi_h = \frac{x_h-x_1}{l_e} \quad (0 \leq \xi_h \leq 1 \leq h \leq n + 1) \quad (14)$$

The unknown field function $w(\xi)$ can be expressed by BSWI scaling functions as

$$w(\xi) = \varphi(a_e) \quad (15)$$

where $\varphi$ and $a_e$ are

$$\begin{align*}
a^e &= \left\{ a_{m,-m+1}^j, a_{m,-m+2}^j, \ldots, a_{m,2/\xi}^j \right\}^T \quad (16) \\
\varphi &= \left\{ \phi_{m,-m+1}(\xi), \phi_{m,-m+2}(\xi), \ldots, \phi_{m,2/\xi}(\xi) \right\} \quad (17)
\end{align*}$$

The element physical DOFs are defined by

$$w^e = \{ w_1, \theta_1, w_2, w_3, \ldots, w_{r-1}, w_r, w_{r+1}, \theta_{r+1} \}^T \quad (18)$$

where $\theta_1 = \frac{1}{l_e} \frac{dw(x_1)}{d\xi}$ and $\theta_{r+1} = \frac{1}{l_e} \frac{dw(x_{r+1})}{d\xi}$. Substituting Eq. (15) into Eq. (18), we obtain

$$w^e = R^e_a a^e \quad (19)$$

where the matrix $R^e_a$ is

$$R^e_a = \left[ \varphi^T(\xi_1), \frac{1}{l_e} \frac{d\varphi^T(\xi_1)}{d\xi}, \varphi^T(\xi_2), \frac{1}{l_e} \frac{d\varphi^T(\xi_2)}{d\xi}, \ldots, \varphi^T(\xi_{r+1}), \frac{1}{l_e} \frac{d\varphi^T(\xi_{r+1})}{d\xi} \right] \quad (20)$$

The element transformation matrix $T^e_\xi$ is defined as

$$T^e_\xi = (R^e_a)^{-1} \quad (21)$$

The wavelet-based shape functions $N^e_\xi$ can be given by

$$N^e_\xi = \varphi T^e_\xi \quad (22)$$

Under the classical assumptions of beam-bending theory, the generalized function of potential energy for Euler beam-bending element is (Wang 2002)

$$\Pi_P = \int_a^b \left\{ \frac{EI}{2} \left( -\frac{d^2w}{dx^2} \right)^2 + q(x)w \right\} dx - \sum_i P_i w(x_i) + \sum_i M_i \left( \frac{d^2w}{dx^2} \right)^2 \quad (23)$$

$w(\xi)$ can be interpolated by the BSWIm, scaling functions as

$$w(\xi) = \varphi T_\xi w^e \quad (24)$$

where $w^e$ and $T^e_\xi$ are shown in Eqs. (18) and (21). Mapping Eq. (23) to element standard solving domain and submitting Eq. (24) into Eq. (23), according to variational principle, let $\delta\Pi_P = 0$, we can obtain element solving equations

$$K^e w^e = P_w^e + P_{w_1}^e + P_{w_2}^e \quad (25)$$

Where

$$\begin{align*}
K^e &= \frac{EI}{l_e} \int_0^1 (T^e_\xi)^T \left( \frac{d^2\varphi^T}{d\xi^2} \right) \left( \frac{d^2\varphi^T}{d\xi^2} \right) T^e_\xi d\xi \\
P_w^e &= (T^e_\xi)^T l_e \int_0^1 \varphi(\xi) \frac{d\varphi^T}{d\xi} d\xi \\
P_{w_1}^e &= \sum_i P_i (T^e_\xi)^T \varphi^T(\xi_i) \\
P_{w_2}^e &= \sum_i P_i (T^e_\xi)^T \varphi^T(\xi_i) \quad (26) \\
P_{w_2}^e &= \sum_i P_i (T^e_\xi)^T \varphi^T(\xi_i) \quad (27) \\
P_{w_2}^e &= \sum_i P_i (T^e_\xi)^T \varphi^T(\xi_i) \quad (28)
\end{align*}$$
For the modal analysis, the vibration equation can be given as

$$(K_e - \omega^2 M_e) \phi = 0$$  \hfill (29)$$

where $\omega$ is the angular frequency, the element stiffness matrix is given by Eq.(26) and element mass matrix is

$$M_e = \int_{\xi} \rho A \left(\frac{\partial \phi}{\partial \xi}\right)^T \phi \ T_e \ d\xi$$  \hfill (30)$$

where $\rho$ is the density, $A$ is the area of the cross-section. To deal with the boundary conditions, the corresponding DOFs are set to zero and eliminated from the equations.

**RESULTS**

To verify the implementation of the proposed method, the vibration of frame structure is analyzed in this paper. The material used in these problems is steel. Its elastic modulus $E = 207 \text{ GPa}$ and density $\rho = 7800 \text{ kg/m}^3$. These structures are composed of beam elements. The cross section of cantilever beam problem is 25 cm$^2$ and the corresponding moment inertia is about 52 cm$^4$. The cross section areas of most of the beams for plane structures are 0.806 cm$^2$ and the corresponding moment inerts of the areas are $2.71 \times 10^{-2}$ cm$^4$. First, a cantilever beam of length 1 meter is analyzed to verify the formulation of the method. The exact solution of the radian frequency $\omega$ is

$$\omega = k^2 \pi^2 \frac{EI}{\rho A L^4}$$  \hfill (31)$$

<table>
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<th>Mode No.</th>
<th>Exact</th>
<th>BSWI 3</th>
<th>relative error (%)</th>
<th>BSWI 4</th>
<th>relative error (%)</th>
<th>finite element</th>
<th>relative error (%)</th>
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</table>

The present computed 5 radian frequencies up to levels 3 and 4 BSWI are listed Table 1 and compared with the results obtained by the finite element method. Fig. 3 shows the relative errors for results obtained by the present solution in comparison with those obtained from the finite element method. The first five mode shapes of the problem are shown in Fig. 4.

As the second example, the vibration of a frame structure shown in Fig. 3 with constant cross sections structure is also solved by the present B-spline wavelets elements method. This problem is used to demonstrate the advantage of the spline approximation. The structure is modeled as 6 elements: four vertical elements and two horizontal elements. Since the lengths of the vertical elements are half of the length of horizontal ones, the vertical elements will be stiffer than the horizontal elements. Therefore, the refinement of horizontal elements is twice of that of vertical elements. Another consideration is that the elemental stiffness in axial effect is stronger than that by bending, hence the resolutions for axial and bending response are treated separately. In the present analysis, only the bending resolution is refined in each level. By the multi-level analysis, the model is uniformly refined up to level 4. Some of the computed frequencies from each level are listed in Table 2. The first 4 mode shapes of vibration are also shown in Fig. 5.
CONCLUSION
A compactly supported semi-orthogonal B-spline wavelet on the interval to construct 1D BSWI elements for structural analysis is presented in this paper. The numerical examples showed the good performance of the method. Some advantages of BSWI elements for structural analysis are discussed. As shown in the paper, BSWI elements have higher efficiency and precision than traditional elements in solving vibration problems. It should be noting that no matter what the layout of element nodes are, the BSWI elements will be constructed freely as long as we can ensure the non-singularity of element transform matrix. It can be concluded that BSWI elements are useful tools to deal with structural problems in engineering.

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Fig. 5. First four mode shapes for Example-2